Exc 1 $\quad A = \begin{pmatrix} 1 & 1 \\ \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \qquad Ax = b \qquad \varepsilon = 10^{-10}$

[0.8] a. normal equations $Ax = b \Rightarrow A^T A x = A^T b$

$$A^T A = \begin{pmatrix} 1 & \varepsilon & 0 \\ 1 & 0 & \varepsilon \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} = \begin{pmatrix} 1+\varepsilon^2 & 1 \\ 1 & 1+\varepsilon^2 \end{pmatrix} \approx \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$\varepsilon^2 = 10^{-20} < 10^{-16}$ round-off error

$A^T A$ is singular, hence problem can not be solved using normal equation

[2] b $\quad$ QR factorization of A using Gram-Schmidt

- process to columns of A ; $A = [a_1 \;\; a_2]$

$A = 3 \times 2 \Rightarrow Q = 3 \times 2 \quad R = 2 \times 2$

- $u_1 = a_1 = \begin{pmatrix} 1 \\ \varepsilon \\ 0 \end{pmatrix} \Rightarrow v_1 = \dfrac{u_1}{\|u_1\|} = \dfrac{1}{\sqrt{1+\varepsilon^2}} \begin{pmatrix} 1 \\ \varepsilon \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ \varepsilon \\ 0 \end{pmatrix}$

$$u_2 = a_2 - \langle v_1, a_2 \rangle v_1 = \begin{pmatrix} 1 \\ 0 \\ \varepsilon \end{pmatrix} - \left\langle \begin{pmatrix} 1 \\ \varepsilon \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \varepsilon \end{pmatrix} \right\rangle \begin{pmatrix} 1 \\ \varepsilon \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ \varepsilon \end{pmatrix} - \begin{pmatrix} 1 \\ \varepsilon \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\varepsilon \\ \varepsilon \end{pmatrix} \Rightarrow v_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{2\varepsilon^2}} \begin{pmatrix} 0 \\ -\varepsilon \\ \varepsilon \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$Q = [v_1 \;\; v_2] = \begin{pmatrix} 1 & 0 \\ \varepsilon & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}$$

- $R = \begin{pmatrix} \|u_1\| & \langle v_1, a_2 \rangle \\ 0 & \|u_2\| \end{pmatrix}$

$\|u_1\| = \sqrt{1+\varepsilon^2} \approx 1$

$\|u_2\| = \sqrt{2\varepsilon^2} = \sqrt{2}\,\varepsilon$

$\langle v_1, a_2 \rangle = \left\langle \begin{pmatrix} 1 \\ \varepsilon \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \varepsilon \end{pmatrix} \right\rangle = 1$

$$\Rightarrow R = \begin{pmatrix} 1 & 1 \\ 0 & \sqrt{2}\,\varepsilon \end{pmatrix}$$

- solve $Ax = b \Rightarrow QRx = b \Rightarrow Rx = Q^T b$

R upper triangular and full-rank

$\Rightarrow Rx = Q^T b$ can be solved

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \Rightarrow x_2 = \dots \Rightarrow x_1 = \dots$$

hence, results in solvable system $Rx = Q^T b$

[0.8] 1c singular values $\sigma_i = \sqrt{\lambda_i(A^TA)}$

from a. $A^TA = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\det(A^TA - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda = 0$

$\Rightarrow \lambda_1 = 0$ and $\lambda_2 = 2$

$\Rightarrow$ singular values of A : $\sigma_1 = 0$, $\sigma_2 = \sqrt{2}$

Exc 2 $\quad Ax = b \quad A = \begin{bmatrix} 1 & 10^{-20} \\ 2 & 2.10^{20} \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 2.10^{20} \end{bmatrix}$

$u = 10^{-16}$, use this to round intermediate results

a. solve using GE without pivoting

[0.5]

$$\left[\begin{array}{cc|c} 1 & 10^{-20} & 1 \\ 2 & 2.10^{20} & 2.10^{20} \end{array}\right] \underset{②-2①}{\sim} \left[\begin{array}{cc|c} 1 & 10^{-20} & 1 \\ 0 & 2.10^{20} - 2 10^{-20} & 2.10^{20} - 2 \end{array}\right]$$

$\Rightarrow$ after rounding

$$\left[\begin{array}{cc|c} 1 & 10^{-20} & 1 \\ 0 & 2.10^{20} & 2.10^{20} \end{array}\right] \Rightarrow \begin{array}{l} x_2 = 1 \\ x_1 = 1 - 10^{-20}.1 \approx 1 \end{array}$$

b use partial pivoting

[0.75]

in first column element (2,1) larger than (1,1)

$\Rightarrow$ switch row 2 and 1

$$\left[\begin{array}{cc|c} 2 & 2.10^{20} & 2.10^{20} \\ 1 & 10^{-20} & 1 \end{array}\right] \underset{②-\frac{1}{2}①}{\sim} \left[\begin{array}{cc|c} 2 & 2.10^{20} & 2.10^{20} \\ 0 & 10^{-20} - 10^{20} & 1 - 10^{20} \end{array}\right]$$

$\Rightarrow$ after rounding

$$\left[\begin{array}{cc|c} 2 & 2.10^{20} & 2.10^{20} \\ 0 & -10^{20} & -10^{20} \end{array}\right] \Rightarrow \begin{array}{l} x_2 = 1 \\ x_1 = \frac{1}{2}(2.10^{20} - 2 10^{20}) = \end{array}$$

c use complete pivoting

biggest element in matrix element (2,2), bring this to element (1,1)

[0.75]

first: change row 1 and 2

$$A \rightarrow \begin{bmatrix} 2 & 2.10^{20} \\ 1 & 10^{-20} \end{bmatrix} \qquad b \rightarrow \begin{bmatrix} 2.10^{20} \\ 1 \end{bmatrix}$$

than change order unknowns x_1, x_2 (change columns and

$$\Rightarrow \begin{bmatrix} 2.10^{20} & 2 \\ 10^{-20} & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 2.10^{20} \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2.10^{20} & 2 & 2.10^{20} \\ 10^{-20} & 1 & 1 \end{array}\right] \underset{\sim}{②-\frac{1}{2}10^{-40}①} \left[\begin{array}{cc|c} 2.10^{20} & 2 & 2.10^{20} \\ 0 & 1-10^{-40} & 1-10^{-2} \end{array}\right]$$

$\Rightarrow$ after rounding

$$\left[\begin{array}{cc|c} 2.10^{20} & 2 & 2.10^{20} \\ 0 & 1 & 1 \end{array}\right] \Rightarrow x_1 = 1 \quad , x_2 = (2.10^{20} - 1)/2.10^{20} \approx$$

d partial pivoting with row scaling $\Rightarrow$ maximum on each row is 1

[0.75]

$$A = \begin{bmatrix} 1 & 10^{-20} \\ 2 & 2.10^{20} \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2.10^{20} \end{bmatrix}$$

first row already maximum 1

second row $\times$ $\frac{1}{2} \cdot 10^{-20}$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 10^{-20} & 1 \\ 10^{-20} & 1 & 1 \end{array}\right]$$

no pivoting needed element (1,1) larger than (2,1)

$$\underset{②-10^{-20}①}{\sim} \left[\begin{array}{cc|c} 1 & 10^{-20} & 1 \\ 0 & 1-10^{-40} & 1-10^{-20} \end{array}\right]$$

after rounding

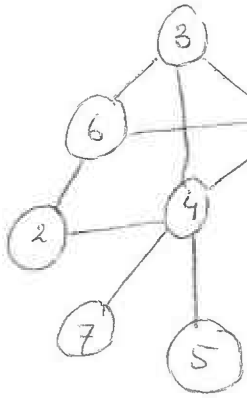
$$\left[\begin{array}{cc|c} 1 & 10^{-20} & 1 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow x_2 = 1 \quad x_1 = 1 - 10^{-20} \simeq 1$$

e. partial pivoting with scaling and complete pivoting give in general the correct result

[0.25]

(in our case: without pivoting gives 'by accident' correct answer)

Exc 3 graph A



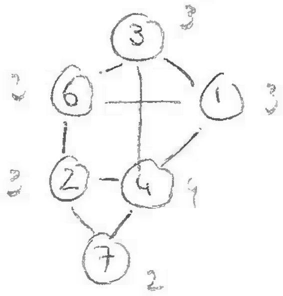
[0.4] a. associated matrix

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} * & 0 & * & * & 0 & * & 0 \\ 0 & * & 0 & * & 0 & * & * \\ * & 0 & * & * & 0 & * & 0 \\ * & * & * & * & * & 0 & * \\ 0 & 0 & 0 & * & * & 0 & 0 \\ * & * & * & 0 & 0 & * & 0 \\ 0 & * & 0 & * & 0 & 0 & * \end{bmatrix} \end{array}$$

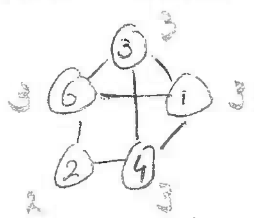
[12] b determine minimum degree ordering and sketch matrix after reordering

- remove vertex with lowest degree, i.e. 5

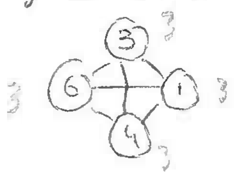
$\Rightarrow$



- remove now vertex with lowest degree, i.e 7



- remove now vertex with lowest degree, i.e 2
  4 and 6 are indirectly related via 2
  removing 2 will introduce connection between 4 - 6



- remaining vertices fully connected with each other
  $\Rightarrow$ their order is not relevant

$\{5, 7, 2, 1, 3, 4, 6\}$

$$\tilde{A} = \begin{array}{c|c} & \begin{array}{ccccccc} 5 & 7 & 2 & 1 & 3 & 4 & 6 \end{array} \\ \begin{array}{c} 5 \\ 7 \\ 2 \\ 1 \\ 3 \\ 4 \\ 6 \end{array} & \begin{bmatrix} * & & & & & * & \\ & * & * & & & * & \\ & * & * & & & * & * \\ & & & * & * & * & * \\ & & & * & * & * & * \\ * & * & * & * & * & * & \\ & & * & * & * & & * \end{bmatrix} \end{array}$$

[0.8] e. sketch the L $\Rightarrow$ fill will occur within convex hull envelope of $\tilde{A}$

$$L = \begin{bmatrix} * & & & & & * & \\ & * & * & & & * & \\ & * & * & & & * & * \\ & & & * & * & * & * \\ & & & * & * & * & * \\ * & * & * & * & * & * & \\ & & * & * & * & * & * \end{bmatrix}$$